TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 3&4

Section One: Calculator-free

SOLUTIONS

WA student number: In figures ☐☐ ☐☐☐ ☐☐☐

In words ______________________________

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Number of additional answer booklets used (if applicable): ☐

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	49	35
Section Two: Calculator-assumed	12	12	100	91	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (49 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(8 marks)**

The equations of planes Π_1, Π_2 and Π_3 are shown below.

Π_1	Π_2	Π_3
$\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$	$\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 18$	$x - y + z = 15$

(a) Show that the equation of plane Π_1 in Cartesian form is $2x + y - z = 6$. (3 marks)

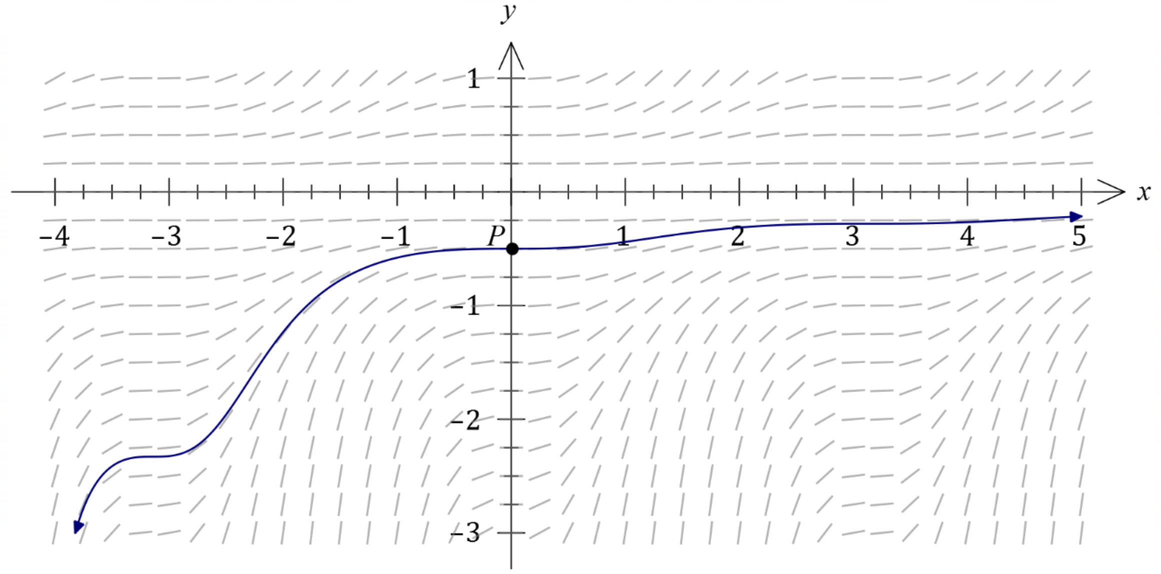
Solution
Cross product of vectors in plane is $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -5 \end{pmatrix}$.
Let normal to plane be $\mathbf{n} = \frac{1}{5}\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 4 + 5 - 3 = 6$.
Hence Cartesian equation is $2x + y - z = 6$.
Specific behaviours
✓ writes cross product of two vectors that lie in Π_1
✓ obtains normal to plane
✓ uses vector equation $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ to derive Cartesian equation

(b) The origin lies on the surface of sphere S. Determine the vector equation of S, given that its centre is the point of intersection of the three planes. (5 marks)

Solution
$2x + y - z = 6$ $2x + y + 2z = 18$ $x - y + z = 15$
Subtracting first from middle gives $3z = 12 \Rightarrow z = 4$. Adding first to last gives $3x = 21 \Rightarrow x = 7$. Hence $y = 7 + 4 - 15 = -4$ and so intersect at $(7, -4, 4)$.
Distance from centre of S to O is $\sqrt{7^2 + 4^2 + 4^2} = 9$.
Hence equation of S is $\left\| \mathbf{r} - \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix} \right\| = 9$.
Specific behaviours
✓ writes system of equations, correctly starts elimination
✓ solves for one variable
✓ solves for all variables
✓ determines distance from origin
✓ correctly writes equation of sphere

Question 2 **(6 marks)**

The point $P(0, -0.5)$ and the slope field given by $\frac{dy}{dx} = y^2 \sin^2 x$ is shown below.



(a) Draw the solution curve through P on the slope field above. (2 marks)

Solution
See slope field
Specific behaviours
✓ curve to left of y-axis has two inflection points and follows slope field ✓ curve to right of y-axis has two inflection points and follows slope field

(b) Determine the equation of the solution curve through P in the form $y = f(x)$. (4 marks)

Solution

$$\int y^{-2}\,dy = \int \sin^2 x\,dx$$

$$\int y^{-2}\,dy = \int \frac{1}{2}(1 - \cos 2x)\,dx$$

$$-\frac{1}{y} = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

$$P(0, -0.5) \Rightarrow c = 2$$

$$y = -\frac{1}{\frac{1}{2}x - \frac{1}{4}\sin 2x + 2} = \frac{4}{\sin 2x - 2x - 8}$$

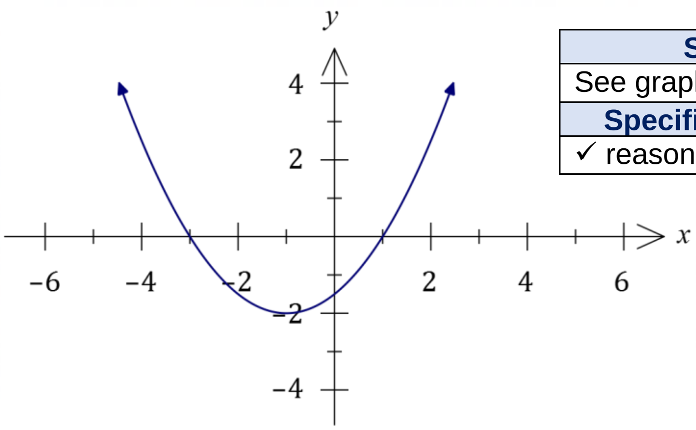
Specific behaviours

✓ separates variables as an integration statement
✓ correctly uses trig identity on RHS
✓ correctly obtains antiderivative, with constant
✓ evaluates constant and writes in required form

Question 3 **(7 marks)**

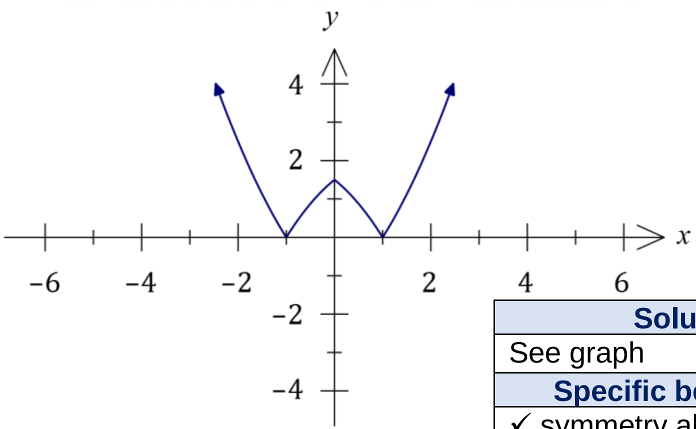
Let $f(x) = \dfrac{(x-1)(x+3)}{2}$.

(a) Sketch the graph of $y = f(x)$. (1 mark)



Solution
See graph
Specific behaviours
✓ reasonable sketch

(b) On the axes below, sketch the graph of $y = |f(|x|)|$. (2 marks)



Solution
See graph
Specific behaviours
✓ symmetry about y-axis
✓ correct sketch

Consider $g(x) = \dfrac{2}{x^2 + 2x - 3}$.

(c) Briefly describe how the graph of $y = f(x)$ can be used to sketch the graph of $y = g(x)$ and hence state the domain and range of $g(x)$. (4 marks)

Solution
The graph of $y = g(x)$ is the same as the graph of $y = \dfrac{1}{f(x)}$.   Domain $D_g = \{x \mid x \in \mathbb{R}, x \neq 1, x \neq -3\}$.   Range $R_g = \left\{y \mid y \in \mathbb{R}, y > 0 \cup y \leq -\frac{1}{2}\right\}$.
Specific behaviours
✓ explains $g(x)$ is reciprocal of $f(x)$ ✓ correct restrictions on x for domain   ✓ states $y > 0$ component of range   ✓ states $y \leq -\frac{1}{2}$ component of range

Question 4 **(7 marks)**

An electronic circuit will remain stable when x and y, the resistances in ohms of two variable resistors in the circuit, satisfy

$$\frac{3}{x}+\frac{5}{y}=\frac{1}{10}.$$

(a) When y is decreasing at a rate of 5 ohms per second, determine the rate that x must be changing for the circuit to remain stable when $y = 200$ ohms. (4 marks)

Solution
Require $\frac{dx}{dt}$ given $\frac{dy}{dt}=-5$, so implicitly differentiate wrt t: $-\frac{3}{x^2}\frac{dx}{dt}-\frac{5}{y^2}\frac{dy}{dt}=0 \Rightarrow \frac{dx}{dt}=-\frac{5x^2}{3y^2}\frac{dy}{dt}$ $\frac{3}{x}+\frac{5}{200}=\frac{1}{10}\rightarrow\frac{3}{x}=\frac{2}{20}-\frac{1}{20}\rightarrow x=40$ $\frac{dx}{dt}=-\frac{5\times 40\times 40}{3\times 200\times 200}\times -5=\frac{1}{3}$ Hence x must increase at $\frac{1}{3}$ ohms per second for stability.
Specific behaviours
✓ implicitly differentiates equation   ✓ obtains expression for $\dot{x}$ ✓ indicates correct values for $\dot{y}$ and x ✓ correctly evaluates $\dot{x}$

(b) The circuit is stable, the resistance of x is increasing at 6 ohms per second and has just reached 60 ohms. Use the technique of increments to calculate the approximate change in the resistance of y in the next tenth of a second. (3 marks)

Solution
$\delta t=0.1,\quad \frac{3}{60}+\frac{5}{y}=\frac{1}{10}\rightarrow\frac{5}{y}=\frac{2}{20}-\frac{1}{20}\rightarrow y=100$ $\delta y\approx\frac{dy}{dt}\delta t$ $=-\frac{3y^2}{5x^2}\frac{dx}{dt}\delta t$ $=-\frac{3\times 100\times 100}{5\times 60\times 60}\times 6\times 0.1$ $=-1\text{ ohm}$ Resistance of y will decrease by 1 ohm.
Specific behaviours
✓ states correct values for y and δt ✓ forms correct expression for δy ✓ correct value for δy

Question 5 **(6 marks)**

Let $f(z) = 2z^3 + az^2 + 12z + b$, where a and b are real constants.

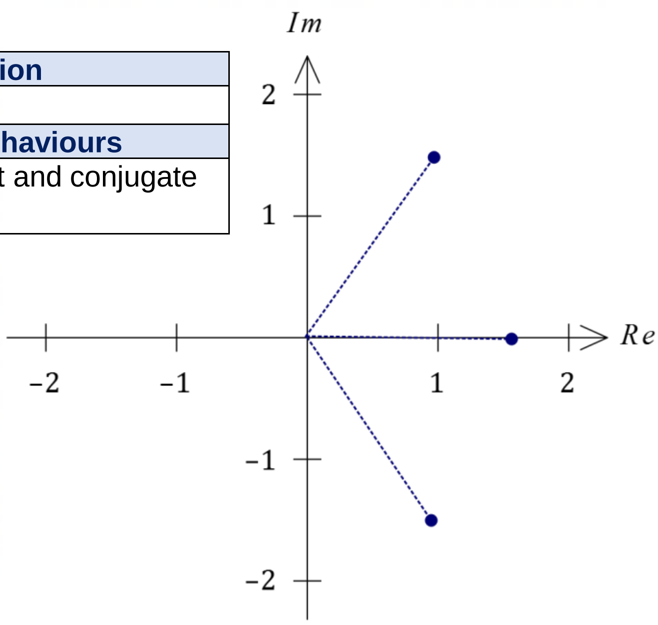
One of the roots of $f(z)$ is $z = 1 - \sqrt{2}i$.

(a) Determine the value of the constant a and the value of the constant b. (4 marks)

Solution
Product of factors using given root and conjugate: $$\left(z-(1-\sqrt{2}i)\right)\left(z-(1+\sqrt{2}i)\right) = z^2 - 2z + 3$$ Hence $$f(z) = 2z^3 + az^2 + 12z + b = (z^2 - 2z + 3)(2z + k)$$ Comparing z coefficients: $$12 = 6 - 2k \Rightarrow k = -3$$ Hence from expansion of factors, $a = k - 4 = -7$ and $b = 3k = -9$. $$a = -7, \qquad b = -9$$
Specific behaviours
✓ indicates conjugate is another root ✓ obtains quadratic factor of $f(x)$ ✓ uses quadratic factor to obtain linear factor ✓ states correct values for a and b

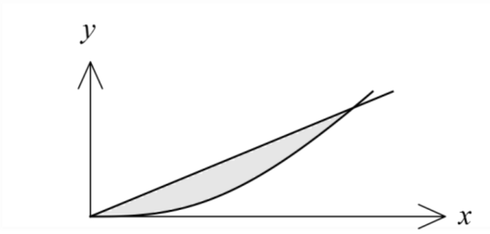
(b) Show all the roots of $f(z)$ in the complex plane below. (2 marks)

Solution
See diagram
Specific behaviours
✓ shows given root and conjugate ✓ shows real root



Question 6 **(8 marks)**

(a) The line $y = x$ and the curve $y = \dfrac{2x^3}{x^2+4}$ are shown in the diagram. They intersect at the origin and at $x = 2$.



Determine the area between the curve and the line in the first quadrant.

Hint: use algebraic long division or substitution (5 marks)

Solution
To determine area under curve let $u = x^2 + 4$, so that $du = 2x\,dx$, $x = 0 \to u = 4$ and $x = 2 \to u = 8$. Then $$\int_0^2 \frac{x^2}{x^2+4}\,2x\,dx = \int_4^8 \frac{u-4}{u}\,du = \int_4^8 1 - \frac{4}{u}\,du = [u - 4\ln u]_4^8 = (8 - 4\ln 8) - (4 - 4\ln 4) = 4 - 4(\ln 8 - \ln 4) = 4 - 4\ln 2$$ Area under line is $\frac{1}{2}(2)(2) = 2$. Hence area between is $2 - (4 - 4\ln 2) = 4\ln(2) - 2$ sq units.
Specific behaviours
✓ chooses suitable substitution ✓ writes integral in terms of u ✓ obtains antiderivative ✓ obtains area under curve ✓ obtains area between

(b) Given that $\displaystyle\int_2^8 \frac{f(x)}{x^2}\,dx = 6$, determine $\displaystyle\int_1^4 f\left(\frac{8}{x}\right)dx$. (3 marks)

Hint: use substitution.

Solution
Let $u = \dfrac{8}{x}$, so that $x = \dfrac{8}{u}$, $dx = -\dfrac{8}{u^2}du$, $x = 1 \to u = 8$ and $x = 4 \to u = 2$. Then $$\int_1^4 f\left(\frac{8}{x}\right)dx = \int_8^2 f(u)\left(-\frac{8}{u^2}\right)du = -(-8)\int_2^8 \frac{f(u)}{u^2}\,du = 8 \times 6 = 48$$
Specific behaviours
✓ chooses substitution and relates dx and du ✓ writes integral in terms of u ✓ derives correct value

Question 7 **(7 marks)**

(a) Determine the value of the constant A and the value of the constant B so that

$$\frac{4}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}.$$

(2 marks)

Solution
Combining RHS into single fraction and equating numerators: $Ax + A + 3Bx - B = 4 \Rightarrow A - B = 4, A + 3B = 0$ Hence $A = 3$ and $B = -1$.
Specific behaviours
✓ indicates appropriate method   ✓ solves for A, B correctly

(b) Hence use the substitution $u = \cos x$ to determine

$$\int \frac{4\sin x}{\cos^2 x + \cos 2x + 2\cos x}\,dx.$$

(5 marks)

Solution
Simplify denominator $\cos^2 x + (2\cos^2 x - 1) + 2\cos x = 3\cos^2 x + 2\cos x - 1$ $= 3u^2 + 2u - 1$ $= (3u-1)(u+1)$ Also $u = \cos x \Rightarrow du = -\sin x\,dx$ Hence $\int \frac{-4}{3\cos^2 x + 2\cos x - 1}(-\sin x)dx = \int \frac{-4}{(3u-1)(u+1)}du$ $= \int \frac{1}{u+1} - \frac{3}{3u-1}du$ $= \ln\lvert u+1\rvert - \ln\lvert 3u-1\rvert + c$ $= \ln\lvert\cos x + 1\rvert - \ln\lvert 3\cos x - 1\rvert + c$ $= \ln\left\lvert\frac{\cos x + 1}{3\cos x - 1}\right\rvert + c$
Specific behaviours
✓ simplifies denominator in terms of $\cos x$ ✓ correctly relates dx and du ✓ uses partial fractions to express integral in terms of u ✓ obtains antiderivative with absolute value brackets   ✓ correct result in terms of x, and integration constant

End of questions

Supplementary page

Question number: _________